Quantitative Equational Reasoning

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- Reasoning about programs is also based on program equivalences.
- The dawning of the age of quantitative reasoning.
- We want quantitative analogues of algebraic reasoning.
- (Pseudo)metrics instead of equivalence relations.
- Equality indexed by a real number $=_\epsilon$.

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- Definitely not an equivalence relation;
- it defines a *uniformity* (but we won't stress this point of view).
- Quantitative analogue of equational reasoning.
- completeness results, universality of free algebras, Birkhoff-like variety theorem, monads ....

Finitary equational theories

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- Theories: set of equations closed under deduction.

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- Everything has finite arity.
- As Ω-algebra $\mathcal{A}$ is a set A to interpret the basic sort and, for each operation f of arity n a function $f_{\mathcal{A}} : A^n \longrightarrow A$.

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- The elements of X are in TX.
- If $t_1, \ldots, t_n$ are in TX and f has arity n then $f(t_1, \ldots, t_n)$ is in TX.

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- Let S be a set of equations between pairs of terms in TX. We define a *congruence relation* $\sim_S$ on TX in the evident way.

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- Easy to check that if $t_1 \sim_S s_1, \ldots, t_n \sim_S s_n$ then $f(t_1, \ldots, t_n) \sim_S f(s_1, \ldots, s_n)$ we can define $f_{\sim_S}$ on $TX/\sim_S$.

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- A class of Ω-algebras satisfying a set of equations is called a variety of algebras (not the same as an algebraic variety!).
- When are a set of equations bad? If we can derive $x = y$ from S then the only algebras have one element.

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- Vector spaces have two sorts.
- Fields are annoying because we have to say $x \neq 0$ implies x^{-1} exists. Fields do not form an equational variety.
- Sometimes we need to state conditional equations; these are called *Horn clauses*. Example: cancellative monoids, $x \cdot y = x \cdot z \vdash y = z$.

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- Axioms:

$$(B_1) \vdash t +_1 t' = t$$
$$(B_2) \vdash t +_\epsilon t = t$$
$$(SC) \vdash t +_\epsilon t' = t' +_{1-\epsilon} t$$
$$(SA) \vdash (t +_\epsilon t') +_{\epsilon'} t'' = t +_{\epsilon\epsilon'} (t' +_{\frac{\epsilon' - \epsilon\epsilon'}{1-\epsilon\epsilon'}} t'')$$

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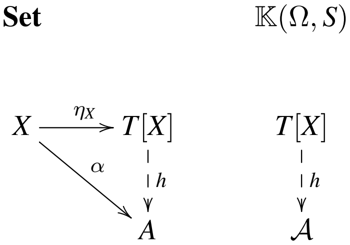

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Example

Consider $\mathbb{Z}_2 \times \mathbb{Z}_2$. It's not a field because, *e.g.* $(1, 0) \times (0, 1) = (0, 0)$. Hence fields cannot be described by equations!

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- Quantitative inferences: $\mathcal{E}(\mathbb{T}X) = \mathcal{P}_{\mathsf{fin}}(\mathcal{V}(\mathbb{T}X)) \times \mathcal{V}(\mathbb{T}X)$

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(Assumpt) If $\phi \in \Gamma$, then $\Gamma \vdash \phi$.

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- Equational theory: $\mathcal{U} = \ \vdash_S \bigcap \mathcal{E}(\mathbb{T}X)$.

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- Morphisms are Ω-algebra homomorphisms that are nonexpansive.
- $\mathbb{T}X$ is an Ω-algebra. $\sigma : \mathbb{T}X \longrightarrow A$, Ω-homomorphism.

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- We write $\{s_i =_{\varepsilon_i} t_i / i = 1, \ldots, n\} \models_{\mathcal{A}} s =_\varepsilon t$.
- We write $\mathbb{K}(\mathcal{U}, \Omega)$ for the algebras satisfying $\mathcal{U}$.

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- If we take the quotient we get an (extended) metric space.
- The resulting algebra is in $\mathbb{K}(\Omega, \mathcal{U})$.
- We can do this for any set M of generators and produce a "free" quantitative algebra.

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- The proof needs to deal with quantitative aspects and uses the infinitary axiom.

Free construction from a metric space

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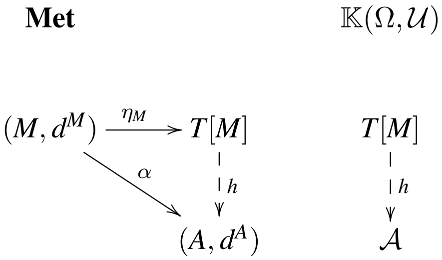
- Given any $\alpha : M \longrightarrow A$ non-expansive we can turn $\mathcal{A} = (A, d)$ into an algebra in $\mathbb{K}(\Omega_M, \mathcal{U}_M)$ by interpreting each $m \in M$ as $\alpha(m) \in A$.

Universal property



$\mathcal{U}_M$ is consistent if and only if the map η_M is an isometry.

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- The total variation metric on probability distributions.

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- $\mathcal{C}(p, q)$ is the set of couplings for (p, q).

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- Splitting lemma: If p, q are Borel probability measures on M and $e = T(p,q)$. There are p', q', r such that

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- The axioms give rise to the total-variation metric.

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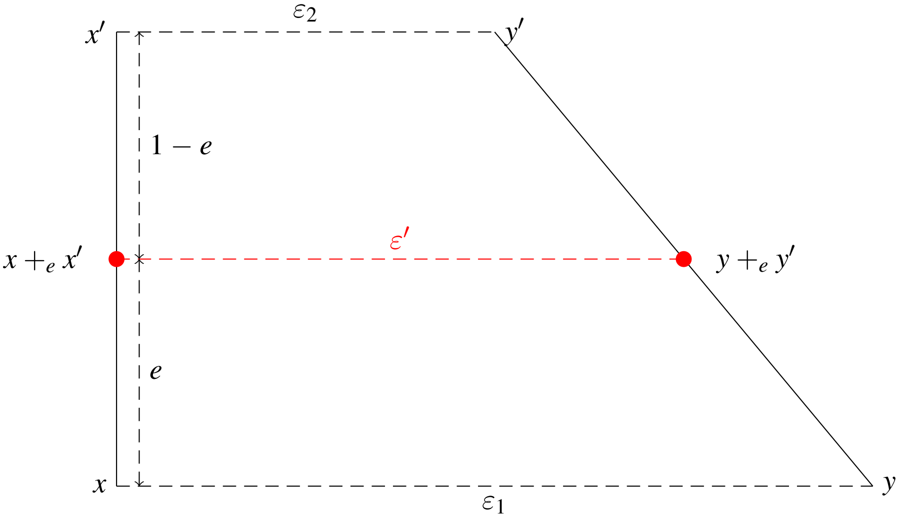
Picture of IB_1



Kantorovich (Wasserstein) metric

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- How do we lift it to the continuous case?

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- The finitely supported probability measures are *dense* in the space of all probability measures.

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- In this case we get a monad on $\mathbf{CSMet}_1$: complete separable 1-bounded metric spaces.

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- Recent work: Variety theorems (LICS 2017), Markov processes by combining theories (LICS 2018), Fixed-point operators (2020), Tensor of theories (2020)